

# NON COMMUTATIVE METRICS ON QUANTUM FAMILIES OF MAPS

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ABSTRACT. We show that any quantum family of maps from a non commutative space to a compact quantum metric space has a canonical quantum semi metric structure.

## 1. INTRODUCTION

One of the basic ideas of Non Commutative Geometry is that any unital  $C^*$ -algebra  $A$  can be considered as the algebra of functions on a *symbolic* compact quantum (non commutative) space  $\mathfrak{Q}A$ . From this point of view, any unital  $*$ -homomorphism  $\Phi : B \longrightarrow A$  between unital  $C^*$ -algebras will be a map  $\mathfrak{Q}\Phi$ , from  $\mathfrak{Q}A$  to  $\mathfrak{Q}B$ . There are many notions in Topology and Geometry that can be translate into NC language:

The notion of *quantum family of maps*, defined by Woronowicz [11] and Soltan [10], conclude from the following fact:

”Every map  $f$  from  $X$  to the set of all maps from  $Y$  to  $Z$  (or in the other word, any *family* of maps from  $Y$  to  $Z$  *parameterized* by  $f$  with *parameters*  $x$  in  $X$ ) can be considered as a map

$$\tilde{f} : X \times Y \longrightarrow Z,$$

defined by  $\tilde{f}(x, y) = f(x)(y)$ .”

**Definition 1.1.** *Let  $B, C$  be unital  $C^*$ -algebras. A quantum family of morphisms from  $B$  to  $C$  (or, a quantum family of maps from  $\mathfrak{Q}C$  to  $\mathfrak{Q}B$ ) is a pair  $(A, \Phi)$  consisting of a unital  $C^*$ -algebra  $A$  and a unital  $*$ -homomorphism  $\Phi : B \longrightarrow C \otimes A$ , where  $\otimes$  denotes the spatial tensor product of  $C^*$ -algebras.*

For more details on quantum families, see [11],[10], and [3].

Another concept that can be translate from Geometry into NC Geometry, is *distance* or *metric*. Marc Rieffel has developed the notion of quantum metric space in a series of papers [5], [6], [7], [8], [9]. Using the *order unite spaces*, he has constructed a general framework for the topic, but in this paper, we deals with special examples of Rieffel’s quantum metric spaces, stated in the  $C^*$ -algebra literature.

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The aim of this note is to show that any quantum family of maps from a quantum space to a compact quantum metric space has a canonical quantum semi metric structure. We are motivated by the following trivial fact:

*Let  $(Z, d)$  be a metric space and  $f : X \times Y \longrightarrow Z$  be a family of maps from  $Y$  to  $Z$ , then  $X$  has a semi metric  $\rho$  defined by*

$$\rho(x, x') = \sup_{y \in Y} d(f(x, y), f(x', y)).$$

## 2. COMPACT QUANTUM SEMI METRIC SPACES

Throughout, for any topological space  $(X, \tau)$  (resp. semi metric space  $(X, d)$ )  $\mathbf{C}(X, \tau)$  (resp.  $\mathbf{C}(X, d)$ ) denotes the  $\mathbf{C}^*$ -algebra of all continuous bounded complex valued maps on  $X$  with the uniform norm. For a semi metric  $d$ ,  $\tau_d$  denotes the topology induced by  $d$ . Let  $(X, d)$  be a semi metric space. For every  $f \in \mathbf{C}(X, d)$ , the Lipschitz semi norm  $\|f\|_d$  is defined by

$$\|f\|_d = \sup\left\{\frac{|f(x) - f(x')|}{d(x, x')} : x, x' \in X, d(x, x') \neq 0\right\}.$$

Also, the Lipschitz algebra of  $(X, d)$  is defined by,

$$\mathbf{Lip}(X, d) = \{f \in \mathbf{C}(X, d) : \|f\|_d < \infty\}.$$

We need the following simple lemma.

**Lemma 2.1.** *Let  $(X, d)$  be a semi metric space and  $a$  be a complex valued map on  $X$ . Then  $a \in \mathbf{Lip}(X, d)$  and  $\|a\|_d \leq 1$  if and only if  $|a(x) - a(x')| \leq d(x, x')$  for every  $x, x' \in X$ . In particular, if  $b \in \mathbf{C}(X, d)$ , then  $\|b\|_d = 0$  if and only if  $b$  is a constant map.*

*Proof.* Let  $a \in \mathbf{Lip}(X, d)$  and  $\|a\|_d \leq 1$ . Suppose that  $x, x' \in X$ . If  $d(x, x') = 0$ , then  $a(x) = a(x')$ , since  $a$  is continuous with  $\tau_d$ . If  $d(x, x') \neq 0$ , then  $1 \geq \|a\|_d \geq \frac{|a(x) - a(x')|}{d(x, x')}$ , and thus  $|a(x) - a(x')| \leq d(x, x')$ . The other direction is trivial.  $\square$

For any  $\mathbf{C}^*$ -algebra  $\mathfrak{A}$ ,  $S(\mathfrak{A})$  denotes the state space of  $\mathfrak{A}$  with  $w^*$  topology. If  $\mathfrak{A}$  is unital,  $1_{\mathfrak{A}}$  denotes the unit element of  $\mathfrak{A}$ .

Let  $\mathcal{A}$  be a self adjoint linear subspace of the  $\mathbf{C}^*$ -algebra  $\mathfrak{A}$ , and let  $L : \mathcal{A} \longrightarrow [0, \infty)$  be a semi norm on  $\mathcal{A}$ . Connes has pointed out [1], [2], that one can define a semi metric  $\rho_L$  on  $S(\mathfrak{A})$  by

$$(1) \quad \rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{A}, L(a) \leq 1\} \quad (\mu, \nu \in S(\mathfrak{A})).$$

Note that  $\rho_L$  can take values  $+\infty$  and 0 for different states of  $\mathfrak{A}$ . Conversely, let  $d$  be a semi metric on  $S(\mathfrak{A})$  (such that the topology induced by  $d$  on  $S(\mathfrak{A})$  is not necessarily  $w^*$  topology). Define a semi norm  $L_d : \mathfrak{A} \longrightarrow [0, +\infty]$  by

$$L_d(a) = \sup\left\{\frac{|\mu(a) - \nu(a)|}{d(\mu, \nu)} : \mu, \nu \in S(\mathfrak{A}), d(\mu, \nu) \neq 0\right\} \quad (a \in \mathfrak{A}).$$

Note that  $L_d(a) = L_d(a^*)$  for every  $a \in \mathfrak{A}$ .

Let  $(X, d)$  be a compact metric space. Consider the Lipschitz semi norm

$$\|\cdot\|_d : \mathbf{Lip}(X, d) \subset \mathbf{C}(X, d) \longrightarrow [0, +\infty).$$

Then it is easily checked that the semi norm  $\rho_{\|\cdot\|_d}$  on the state space of  $\mathbf{C}(X, d)$  is a metric, called Monge-Kantorovich metric [4]. It is well known that the topology induced by  $\rho_{\|\cdot\|_d}$ , is the  $w^*$  topology, and for every  $x, y \in X$ ,  $d(x, y) = \rho_{\|\cdot\|_d}(\delta_x, \delta_y)$ , where  $\delta : X \longrightarrow \mathbf{C}(X, d)^*$  is the point mass measure map.

**Proposition 2.2.** *Let  $(X, \tau)$  be a compact Hausdorff space and  $d$  be a semi metric on  $X$  such that the topology induced by  $d$  on  $X$  is weaker than  $\tau$ , i.e.  $\tau_d \subset \tau$ . Consider the Lipschitz semi norm  $\|\cdot\|_d : \mathbf{Lip}(X, d) \subset \mathbf{C}(X, \tau) \longrightarrow [0, +\infty)$  and let  $\rho = \rho_{\|\cdot\|_d}$ . Then the following are satisfied.*

- i)  $d(x, y) = \rho(\delta_x, \delta_y)$ , for every  $x, y \in X$ .
- ii)  $L_\rho = \|\cdot\|_d$  on  $\mathbf{C}(X, d) \subset \mathbf{C}(X, \tau)$ .
- iii) Let  $a \in \mathbf{C}(X, \tau)$ , then  $a \in \mathbf{C}(X, d)$  if and only if the map  $\nu \longmapsto \nu(a)$  on  $S(\mathbf{C}(X, \tau))$  is continuous with  $\rho$ .
- iv) the topology induced by  $\rho$  on  $S(\mathbf{C}(X, \tau))$  is weaker than the  $w^*$  topology.

*Proof.* i) Let  $x, y$  be in  $X$ . Suppose that  $a \in \mathbf{Lip}(X, d)$  and  $\|a\|_d \leq 1$ . Then by Lemma 2.1,  $|\delta_x(a) - \delta_y(a)| = |a(x) - a(y)| \leq d(x, y)$ , and thus by definition of  $\rho$ , we have  $\rho(\delta_x, \delta_y) \leq d(x, y)$ . Conversely, let  $a_x \in \mathbf{C}(X, d)$  be defined by  $a_x(z) = d(x, z)$  ( $z \in X$ ); then for every  $x', y' \in X$ ,  $|a_x(x') - a_x(y')| = |d(x, x') - d(x, y')| \leq d(x', y')$ , and thus by lemma 2.1,  $a_x \in \mathbf{Lip}(X, d)$  and  $\|a_x\|_d \leq 1$ . Now, we have

$$\rho(\delta_x, \delta_y) \geq |\delta_x(a_x) - \delta_y(a_x)| = |a_x(x) - a_x(y)| = d(x, y).$$

ii) By i) and definitions of  $L_\rho$  and  $\|\cdot\|_d$ , it is clear that  $\|\cdot\|_d \leq L_\rho$  on  $\mathbf{C}(X, \tau)$ .

Let  $a \in \mathbf{C}(X, d)$ . If  $\|a\|_d = 0$ , then by Lemma 2.1,  $a$  is a constant map and thus  $L_\rho(a) = 0$ . If  $\|a\|_d = \infty$  then  $L_\rho(a) = \infty$  since  $\|a\|_d \leq L_\rho(a)$ . Thus suppose that  $0 < \|a\|_d < \infty$ . Then for every  $\mu, \nu \in S(\mathbf{C}(X, \tau))$ , we have

$$\rho(\mu, \nu) \geq |\mu(\frac{a}{\|a\|_d}) - \nu(\frac{a}{\|a\|_d})| = \frac{|\mu(a) - \nu(a)|}{\|a\|_d}$$

and thus if  $\rho(\mu, \nu) \neq 0$  then  $\|a\|_d \geq \frac{|\mu(a) - \nu(a)|}{\rho(\mu, \nu)}$ . Therefore,

$$\|a\|_d \geq \sup\left\{\frac{|\mu(a) - \nu(a)|}{\rho(\mu, \nu)} : \mu, \nu \in S(\mathbf{C}(X, \tau)), \rho(\mu, \nu) \neq 0\right\} = L_\rho(a).$$

iii) The "if" part is an immediate consequence of i). For the other direction, we need some notations:

Let  $\sim$  be the equivalence relation on  $X$  defined by

$$x \sim x' \iff d(x, x') = 0.$$

Let  $Y = X/\sim$  and let  $\hat{\cdot} : X \longrightarrow Y$  be the canonical projection. Then  $\hat{d}$ , defined by  $\hat{d}(\hat{x}_1, \hat{x}_2) = d(x_1, x_2)$ , is a well defined metric on  $Y$ , and  $\hat{\cdot}$  is an isometry between  $(X, d)$  and  $(Y, \hat{d})$ . Thus the  $C^*$ -algebras  $\mathbf{C}(X, d)$  and  $\mathbf{C}(Y, \hat{d})$ , and the Lipschitz algebras  $(\mathbf{Lip}(X, d), \|\cdot\|_d)$  and  $(\mathbf{Lip}(Y, \hat{d}), \|\cdot\|_{\hat{d}})$  are isometric isomorph. In particular, the topology induced by  $\rho$  on  $S(\mathbf{C}(X, d))$  is the  $w^*$  topology, since as mentioned above the Monge-Kantorovich metric  $\rho_{\|\cdot\|_{\hat{d}}}$  induces the  $w^*$  topology on  $S(\mathbf{C}(Y, \hat{d}))$ . Consider the canonical embedding  $\Phi : \mathbf{C}(X, d) \longrightarrow \mathbf{C}(X, \tau)$ . For every  $\nu, \nu' \in S(\mathbf{C}(X, \tau))$ ,  $\nu \circ \Phi$  and  $\nu' \circ \Phi$  are in  $S(\mathbf{C}(X, d))$  and

$$(2) \quad \rho(\nu, \nu') = \rho(\nu \circ \Phi, \nu' \circ \Phi).$$

Now, let  $a \in \mathbf{C}(X, d)$  and  $\nu_i \longrightarrow \nu$  be a convergent net in  $S(\mathbf{C}(X, \tau))$  with  $\rho$ . Then  $\nu_i \circ \Phi \longrightarrow \nu \circ \Phi$  is a convergent net in  $S(\mathbf{C}(X, d))$  with  $\rho$ , and since the topology induced by  $\rho$  agrees with the  $w^*$  topology on  $S(\mathbf{C}(X, d))$ , we have

$$\nu_i(a) = \nu_i \circ \Phi(a) \longrightarrow \nu \circ \Phi(a) = \nu(a).$$

Thus we get the desired result.

iv) Let  $\nu_i \longrightarrow \nu$  be a convergent net in  $S(\mathbf{C}(X, \tau))$  with  $w^*$  topology. Thus as in the proof of iii),  $\nu_i \circ \Phi \longrightarrow \nu \circ \Phi$  with  $\rho$ , and by (2),  $\nu_i \longrightarrow \nu$  in  $S(\mathbf{C}(X, \tau))$  with the topology induced by  $\rho$ . This completes the proof of iv).  $\square$

**Definition 2.3.** By a compact quantum semi metric space (QSM space, for short) we mean a triple  $(\mathfrak{A}, \mathcal{A}, L)$ , where  $\mathfrak{A}$  is a unital  $C^*$ -algebra,  $\mathcal{A}$  is a self adjoint linear subspace of  $\mathfrak{A}$  with  $1_{\mathfrak{A}} \in \mathcal{A}$ , and  $L : \mathcal{A} \longrightarrow [0, +\infty)$  is a semi norm such that

- (a)  $L(a) = L(a^*)$  for every  $a \in \mathcal{A}$ ,
- (b) for every  $a \in \mathcal{A}$ ,  $L(a) = 0$  if and only if  $a \in \mathbb{C}1_{\mathfrak{A}}$ , and
- (c) the topology induced by the semi metric  $\rho_L$  on  $S(\mathfrak{A})$  is weaker than the  $w^*$  topology.

As an immediate corollary of the definition, for any compact quantum semi metric space  $(\mathfrak{A}, \mathcal{A}, L)$ , the topology induced by  $\rho_L$  on  $S(\mathfrak{A})$  is compact and in particular the diameter of  $S(\mathfrak{A})$  under  $\rho_L$  is finite.

The following is easily checked.

**Proposition 2.4.** Let  $(\mathfrak{A}, \mathcal{A}, L)$  be a QSM space. Then, for every  $a \in \mathcal{A}$ , the map  $\mu \longmapsto \mu(a)$  on  $S(\mathfrak{A})$  is continuous with topology induced by  $\rho_L$ .

**Definition 2.5.** A QSM space  $(\mathfrak{A}, \mathcal{A}, L)$  is called a compact quantum metric space (QM space, for short) if  $\mathcal{A}$  is a dense subspace of  $\mathfrak{A}$ .

Let  $(\mathfrak{A}, \mathcal{A}, L)$  be a QM space and  $\mu, \nu$  be two different states of  $\mathfrak{A}$ . Then since  $\mathcal{A}$  is dense in  $\mathfrak{A}$ , there is  $a \in \mathcal{A}$  such that  $\mu(a) \neq \nu(a)$ . Thus (by (1))  $\rho_L$  is a metric on  $S(\mathfrak{A})$ . It is an elementary result in Topology that any Hausdorff topology  $\tau$  weaker than a compact Hausdorff topology  $\tau'$  on a

set  $X$ , is equal to the same topology  $\tau'$ . Using this, we conclude that the topology induced by  $\rho_L$  on  $S(\mathfrak{A})$  is the  $w^*$  topology.

**Example 1.** Let  $(X, d)$  be a compact metric space. Then

$$(\mathbf{C}(X, d), \mathbf{Lip}(X, d), \|\cdot\|_d)$$

is a compact quantum metric space.

**Example 2.** Let  $(X, \tau)$  be a compact Hausdorff space and let  $d$  be a semi metric on  $X$  such that  $\tau_d \subset \tau$ . Then Proposition 2.2 and Lemma 2.1, show

$$(\mathbf{C}(X, \tau), \mathbf{Lip}(X, d), \|\cdot\|_d)$$

is a compact quantum semi metric space.

**Remark.** Let  $(\mathfrak{A}, \mathcal{A}, L)$  be a QM space and  $A \subset \mathcal{A}$  be the linear subspace of all self-adjoint elements of  $\mathcal{A}$ . Then  $A$  is an order unite space and  $(A, L|_A)$  is a compact quantum metric space in the sense of Riffel's definition [7].

**Lemma 2.6.** *Let  $\mathfrak{A}$  be a  $C^*$ -algebra with the  $C^*$ -norm  $\|\cdot\|$ ,  $\mathcal{A}$  be a self adjoint linear subspace of  $\mathfrak{A}$  containing  $1_{\mathfrak{A}}$  and  $L : \mathcal{A} \rightarrow [0, +\infty)$  be a semi norm such that for every  $a \in \mathcal{A}$ ,  $L(a) = 0$  if and only if  $a \in \mathbb{C}1_{\mathfrak{A}}$ . Let  $\tilde{L}$  and  $\|\cdot\|$  denote the quotient norm of  $L$  and  $\|\cdot\|$  on  $\frac{\mathcal{A}}{\mathbb{C}1_{\mathfrak{A}}}$  and  $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$ , respectively. Suppose that the image of  $\{a \in \mathcal{A} : L(a) \leq 1\}$  in  $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$  is totally bounded for  $\|\cdot\|$ . Then the topology induced by  $\rho_L$  on  $S(\mathfrak{A})$  is weaker than the  $w^*$  topology.*

*Proof.* See Theorem 1.8 of [5]. □

**Example 3.** Let  $\mathfrak{A}$  be a finite dimensional  $C^*$ -algebra and  $N$  be a Banach space norm on  $\mathfrak{A}$  such that  $N(a) = N(a^*)$  for every  $a \in \mathfrak{A}$ . Let the semi norm  $N_0 : \mathfrak{A} \rightarrow [0, \infty)$  be defined by

$$N_0 = \inf\{N(a + \lambda 1_{\mathfrak{A}}) : \lambda \in \mathbb{C}\}.$$

Since  $\mathfrak{A}$  is finite dimensional, the  $C^*$ -norm of  $\mathfrak{A}$  and  $N$  are equivalent. Thus the image  $K$  of  $\{a \in \mathfrak{A} : N_0(a) \leq 1\}$  is closed and bounded in  $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$ . Again, since  $\mathfrak{A}$  is finite dimensional,  $K$  is compact and thus totally bounded for the quotient norm of the  $C^*$ -norm. Thus by Lemma 2.6,  $(\mathfrak{A}, \mathfrak{A}, N_0)$  is a QM space.

**Example 4.** Let  $G$  be a compact Hausdorff group with identity element  $e$ . Let  $\ell$  be a length function on  $G$ , i.e.  $\ell$  is a continuous non negative real valued function on  $G$  such that

- (i)  $\ell(gg') \leq \ell(g) + \ell(g')$ , for every  $g, g' \in G$ ,
- (ii)  $\ell(g) = \ell(g^{-1})$  for every  $g \in G$ , and
- (iii)  $\ell(g) = 0$  if and only if  $g = e$ .

Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra with a strongly continuous action  $\cdot : G \times \mathfrak{A} \rightarrow \mathfrak{A}$  of  $G$  by automorphisms of  $\mathfrak{A}$ , i.e.

- (a) for every  $g \in G$  the map  $a \mapsto g \cdot a$  is a  $*$ -automorphism of  $\mathfrak{A}$ ,
- (b)  $e \cdot a = a$  for every  $a \in \mathfrak{A}$ ,
- (c)  $g \cdot (g' \cdot a) = (gg') \cdot a$ , for every  $g, g' \in G, a \in \mathfrak{A}$ , and

- (d) if  $g_i \longrightarrow g$  is a convergent net in  $G$  and  $a \in \mathfrak{A}$ , then  $g_i \cdot a \longrightarrow g \cdot a$  with the  $C^*$ -norm of  $\mathfrak{A}$ .

Define a semi norm  $L$  on  $\mathfrak{A}$  by

$$L(a) = \sup\left\{\frac{\|g \cdot a - a\|}{\ell(g)} : g \in G, g \neq e\right\} \quad (a \in \mathfrak{A}).$$

Let  $\mathcal{A} = \{a \in \mathfrak{A} : L(a) < +\infty\}$ . Then by Proposition 2.2 of [5],  $\mathcal{A}$  is a dense  $*$ -subalgebra of  $\mathfrak{A}$ . Now, suppose that the action of  $G$  is ergodic, i.e. if  $a \in \mathfrak{A}$  and for every  $g \in G$ ,  $g \cdot a = a$ , then  $a \in \mathbb{C}1_{\mathfrak{A}}$ . Then it is trivial that  $L(a) = 0$  if and only if  $a \in \mathbb{C}1_{\mathfrak{A}}$ . Rieffel has proved [5, Theorem 2.3], that the topology induced by  $\rho_L$  on  $S(\mathfrak{A})$  agrees with the  $w^*$  topology. Thus  $(\mathfrak{A}, \mathcal{A}, L)$  is a QM space.

For some other examples that completely match our notion of QM space, see [5]. As we will see in the next section, using quantum family of morphisms we can construct many QSM spaces from a QSM space.

### 3. THE MAIN DEFINITION

We need the following simple topological lemma.

**Lemma 3.1.** *Let  $Y$  be a compact space,  $X$  be an arbitrary space and  $(Z, \rho)$  be a semi metric space. Also, let  $\mathbf{C}(Y, Z)$  be the space of all continuous maps from  $Y$  to  $Z$ , with the semi metric  $\hat{\rho}$  defined by*

$$\hat{\rho}(f, g) = \sup\{\rho(f(y), g(y)) : y \in Y\} \quad (f, g \in \mathbf{C}(Y, Z)).$$

*Suppose that  $F : Y \times X \longrightarrow Z$  is a continuous map. Then the map  $\tilde{F} : X \longrightarrow \mathbf{C}(Y, Z)$ , defined by  $\tilde{F}(x)(y) = F(y, x)$  is continuous.*

*Proof.* Let  $x_0 \in X$  and  $\epsilon > 0$  be arbitrary. Since  $F$  is continuous, for every  $y \in Y$ , there are open sets  $U_y, V_y$  in  $X$  and  $Y$  respectively, such that  $(y, x_0) \in V_y \times U_y$  and  $\rho(F(y, x_0), F(y', x)) < \epsilon/2$  for every  $(y', x) \in V_y \times U_y$ . Since  $Y$  is compact, there are  $y_1, \dots, y_n \in Y$  such that  $Y = \cup_{i=1}^n V_{y_i}$ . Let  $W$  be the open set  $\cap_{i=1}^n U_{y_i}$ . Let  $x \in W$  and  $y \in Y$  be arbitrary. Then for some  $i$  ( $i = 1, \dots, n$ ),  $y$  belongs to  $V_{y_i}$  and we have,

$$\rho(F(y, x), F(y, x_0)) \leq \rho(F(y, x), F(y_i, x_0)) + \rho(F(y_i, x_0), F(y, x_0)) < \epsilon.$$

Thus we have  $\hat{\rho}(\tilde{F}(x), \tilde{F}(x_0)) < \epsilon$  for every  $x \in W$ . The proof is complete.  $\square$

Let  $(\mathfrak{A}, \mathcal{A}, L)$  be a QSM space,  $\mathfrak{B}$  be a unital  $C^*$ -algebra, and  $(\mathfrak{C}, \Phi)$  be a quantum family of morphisms from  $\mathfrak{A}$  to  $\mathfrak{B}$ ,  $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B} \otimes \mathfrak{C}$ .

Let  $d$  be a semi metric on  $S(\mathfrak{C})$ , defined by

$$d(\nu, \nu') = \sup\{\rho_L((\mu \otimes \nu)\Phi, (\mu \otimes \nu')\Phi) : \mu \in S(\mathfrak{B})\} \quad (\nu, \nu' \in S(\mathfrak{C})).$$

**Proposition 3.2.** *With the above assumptions, let  $\mathcal{C}$  be the linear space of all  $c \in \mathfrak{C}$  such that the map  $\nu \longmapsto \nu(c)$  on  $S(\mathfrak{C})$  is continuous with the topology induced by  $d$ , and  $L_d(c) < \infty$ . Then the following are satisfied.*

- i)  $\mathcal{C}$  is a self adjoint linear subspace of  $\mathfrak{C}$  and  $1_{\mathfrak{C}} \in \mathcal{C}$ .

- ii) For every  $c \in \mathcal{C}$ ,  $L_d(c) = 0$  if and only if  $c \in \mathbb{C}1_{\mathfrak{C}}$ .
- iii) The topology induced by  $d$  on  $S(\mathfrak{C})$  is weaker than the  $w^*$  topology.
- iv) With the restriction of the domain of  $L_d$  to  $\mathcal{C}$ ,  $\rho_{L_d} \leq d$ .
- v) The topology induced by  $\rho_{L_d}$  on  $S(\mathfrak{C})$  is weaker than the  $w^*$  topology.

*Proof.* i) is easily checked.

ii) Let  $c$  be in  $\mathcal{C}$  and  $L_d(c) = 0$ . By Lemma 2.1, the map  $\nu \mapsto \nu(c)$  on  $S(\mathfrak{C})$  is constant, and thus  $c \in \mathbb{C}1_{\mathfrak{C}}$ .

iii) Apply Lemma 3.1, with  $X = S(\mathfrak{C})$ ,  $Y = S(\mathfrak{B})$ ,  $Z = S(\mathfrak{A})$ ,  $\rho = \rho_L$  and  $F : Y \times X \rightarrow Z$  defined by

$$F(\mu, \nu) = (\mu \otimes \nu)\Phi \quad (\mu \in Y, \nu \in X).$$

We get  $\tilde{F} : X \rightarrow \mathbf{C}(Y, Z)$  is continuous with the metric  $\hat{\rho}$  on  $\mathbf{C}(Y, Z)$ . On the other hand, for every  $\nu, \nu'$  we have  $d(\nu, \nu') = \hat{\rho}(\tilde{F}(\nu), \tilde{F}(\nu'))$ . Thus, if  $\nu_i \rightarrow \nu$  is a convergent net in  $X$  with  $w^*$  topology, then

$$d(\nu_i, \nu) = \hat{\rho}(\tilde{F}(\nu_i), \tilde{F}(\nu)) \rightarrow 0.$$

This implies that the topology induced by  $d$  is weaker than the  $w^*$  topology.

iv) Let  $\nu, \nu'$  be in  $S(\mathfrak{C})$ . If  $d(\nu, \nu') = 0$  then for every  $c \in \mathcal{C}$ ,  $\nu(c) = \nu'(c)$  (since the map  $\mu \mapsto \mu(c)$  is continuous with  $d$ ) and thus by the definition of  $\rho_{L_d}$ ,  $\rho_{L_d}(\nu, \nu') = 0$ . Thus suppose that  $d(\nu, \nu') \neq 0$ . Let  $c \in \mathcal{C}$  with  $L_d(c) \leq 1$ . Then  $1 \geq L_d(c) \geq \frac{|\nu(c) - \nu'(c)|}{d(\nu, \nu')}$ , and thus  $|\nu(c) - \nu'(c)| \leq d(\nu, \nu')$ . Therefore

$$\rho_{L_d}(\nu, \nu') \leq d(\nu, \nu').$$

v) follows directly from iv) and iii).  $\square$

**Definition 3.3.** With the above assumptions, Proposition 3.2, shows that  $(\mathfrak{C}, \mathcal{C}, L_d)$  is a QSM space that is called QSM space induced by the QSM space  $(\mathfrak{A}, \mathcal{A}, L)$  and quantum family of maps  $(\mathfrak{C}, \Phi)$ .

**Lemma 3.4.** With the above assumptions, let  $a \in \mathcal{A}$  and let  $\mu \in S(\mathfrak{B})$ . Then  $c = (\mu \otimes id_{\mathfrak{C}})\Phi(a)$  is in  $\mathcal{C}$ , and  $L_d(c) \leq L(a)$ .

*Proof.* We first show that  $L_d(c) \leq L(a) (< \infty)$ . If  $L(a) = 0$  then  $a \in \mathbb{C}1_{\mathfrak{A}}$  and thus  $c \in \mathbb{C}1_{\mathfrak{C}}$  and  $L_d(c) = 0$ . Suppose that  $L(a) \neq 0$ . We prove that for every  $\nu, \nu' \in S(\mathfrak{C})$  with  $d(\nu, \nu') \neq 0$ ,

$$(3) \quad \frac{|\nu(c) - \nu'(c)|}{d(\nu, \nu')} \leq L(a).$$

Let  $\nu, \nu' \in S(\mathfrak{C})$  be such that  $d(\nu, \nu') \neq 0$ . If  $|\nu(c) - \nu'(c)| = 0$ , then (3) is satisfied. Suppose that

$$|\nu(c) - \nu'(c)| = |(\mu \otimes \nu)\Phi(a) - (\mu \otimes \nu')\Phi(a)| \neq 0.$$

By the definition of  $d$ , we have  $d(\nu, \nu') \geq \rho_L((\mu \otimes \nu)\Phi, (\mu \otimes \nu')\Phi)$ . On the other hand, by the definition of  $\rho_L$ ,

$$\begin{aligned} \rho_L((\mu \otimes \nu)\Phi, (\mu \otimes \nu')\Phi) &\geq |(\mu \otimes \nu)\Phi(\frac{a}{L(a)}) - (\mu \otimes \nu')\Phi(\frac{a}{L(a)})| \\ &= \frac{|(\mu \otimes \nu)\Phi(a) - (\mu \otimes \nu')\Phi(a)|}{L(a)}. \end{aligned}$$

Thus, (3) is satisfied and  $L_d(c) \leq L(a)$ .

Now, we show that the map  $\nu \mapsto \nu(c)$  on  $S(\mathfrak{C})$  is continuous with  $\tau_d$ . Let  $\nu_n \rightarrow \nu$  be a convergent sequence in  $S(\mathfrak{C})$  with the metric  $d$ . Thus, by the definition of  $d$ , we have

$$\rho_L((\mu \otimes \nu_n)\Phi, (\mu \otimes \nu)\Phi) \rightarrow 0.$$

Therefore, by Proposition 2.4,

$$\nu_n(c) = (\mu \otimes \nu_n)\Phi(a) \rightarrow (\mu \otimes \nu)\Phi(a) = \nu(c).$$

□

**Proposition 3.5.** *With the above assumptions, suppose that  $(\mathfrak{A}, \mathcal{A}, L)$  is a QM space and the linear span of*

$$G = \{(\mu \otimes id_{\mathfrak{C}})\Phi(a) : \mu \in S(\mathfrak{B}), a \in \mathfrak{A}\}$$

*is dense in  $\mathfrak{C}$  (for example  $\Phi$  is surjective). Then  $(\mathfrak{C}, \mathcal{C}, L_d)$  is a QM space.*

*Proof.* Since  $\mathcal{A}$  is dense in  $\mathfrak{A}$  and the linear span of  $G$  is dense in  $\mathfrak{C}$ , we have

$$G_0 = \{(\mu \otimes id_{\mathfrak{C}})\Phi(a) : \mu \in S(\mathfrak{B}), a \in \mathcal{A}\}$$

is dense in  $\mathfrak{C}$ . On the other hand, by Lemma 3.4,  $G_0 \subset \mathcal{C}$ . Thus  $\mathcal{C}$  is dense in  $\mathfrak{C}$  and  $(\mathfrak{C}, \mathcal{C}, L_d)$  is a QM space. □

**Example 5.** Let  $\mathfrak{A}$  and  $\mathfrak{C}$  be unital  $C^*$ -algebras. Suppose that  $\mathfrak{A} \otimes \mathfrak{C}$  has a QSM structure. Consider  $*$ -homomorphisms

$$id : \mathfrak{A} \otimes \mathfrak{C} \rightarrow \mathfrak{A} \otimes \mathfrak{C} \quad \text{and} \quad F : \mathfrak{A} \otimes \mathfrak{C} \rightarrow \mathfrak{C} \otimes \mathfrak{A},$$

where  $F$  is the *flip* map, i.e.  $F(a \otimes c) = c \otimes a$  for  $a \in \mathfrak{A}, c \in \mathfrak{C}$ . Then

$$(\mathfrak{C}, id_{\mathfrak{A} \otimes \mathfrak{C}}) \quad \text{and} \quad (\mathfrak{A}, F)$$

are quantum families of morphisms. Thus  $\mathfrak{A}$  and  $\mathfrak{C}$  have naturally QSM structures. Also, by Proposition 3.5, if  $\mathfrak{A} \otimes \mathfrak{C}$  has a QM structure then so are  $\mathfrak{A}$  and  $\mathfrak{C}$ .

**Example 6.** Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra and suppose that  $\mathfrak{A}$  has a QSM structure. Let  $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$  be a unital  $*$ -homomorphism. Then  $(\mathfrak{B}, \Phi)$  can be considered as a quantum family of morphisms from  $\mathfrak{A}$  to  $\mathbb{C}$ . Thus  $\mathfrak{B}$  naturally has a QSM structure. Also, if  $\Phi$  is surjective and  $\mathfrak{A}$  has a QM structure, then by Proposition 3.5,  $\mathfrak{B}$  has a QM structure.

## 4. THE COMMUTATIVE CASE

In this last section we study induced metric structures on ordinary families of maps. The following Lemma is different from part i) of Proposition 2.2 !

**Lemma 4.1.** *Let  $(X, \tau)$  be a compact Hausdorff space and let  $d$  be a semi metric on  $S(\mathbf{C}(X, \tau))$  such that  $\tau_d$  is weaker than the  $w^*$  topology. Let  $\mathcal{C}$  be the space of all  $c \in \mathbf{C}(X, \tau)$  such that the map  $\nu \mapsto \nu(c)$  is continuous on  $S(\mathbf{C}(X, \tau))$  and  $L_d(c) < \infty$ . Consider the semi norm  $L_d : \mathcal{C} \rightarrow [0, +\infty)$ . Then for every  $x, x' \in X$ ,  $d(\delta_x, \delta_{x'}) = \rho_{L_d}(\delta_x, \delta_{x'})$ .*

*Proof.* Let  $x, x'$  be in  $X$ . By the definition of  $\rho_{L_d}$ , we have

$$(4) \quad \rho_{L_d}(\delta_x, \delta_{x'}) = \sup\{|a(x) - a(x')| : a \in \mathcal{C}, L_d(a) \leq 1\}.$$

Let  $a \in \mathcal{C}$  and  $L_d(a) \leq 1$ . If  $d(\delta_x, \delta_{x'}) = 0$ , then  $a(x) = a(x')$  since the map  $\delta_x \mapsto \delta_x(a) = a(x)$  is continuous with  $d$ , thus (4) implies that

$$\rho_{L_d}(\delta_x, \delta_{x'}) = d(\delta_x, \delta_{x'}) = 0.$$

Now, suppose that  $d(\delta_x, \delta_{x'}) \neq 0$ . Since  $1 = L_d(a) \geq \frac{|a(x) - a(x')|}{d(\delta_x, \delta_{x'})}$ , we have  $d(\delta_x, \delta_{x'}) \geq |a(x) - a(x')|$ , thus (4) implies that  $\rho_{L_d}(\delta_x, \delta_{x'}) \leq d(\delta_x, \delta_{x'})$ . Now, define a map  $b_x$  on  $X$  by  $b_x(y) = d(\delta_x, \delta_y)$ . Then  $b_x \in \mathcal{C}$  and  $L_d(b_x) \leq 1$ . Thus

$$\rho_{L_d}(\delta_x, \delta_{x'}) \geq |b_x(x) - b_x(x')| = d(\delta_x, \delta_{x'}).$$

This completes the proof.  $\square$

**Theorem 4.2.** *Let  $(X, \tau)$ ,  $(Y, \tau')$ ,  $(Z, \tau'')$  be compact Hausdorff spaces and let  $d_0$  be a semi metric on  $X$  such that  $\tau_{d_0} \subset \tau$ . Let*

$$F : Y \times Z \rightarrow X$$

*be a continuous map with  $\tau, \tau', \tau''$ , and define a semi metric  $d_1$  on  $Z$  by*

$$d_1(z, z') = \sup_{y \in Y} d_0(F(y, z), F(y, z')).$$

*With the canonical identification  $\mathbf{C}(Y \times Z, \tau' \times \tau'') \cong \mathbf{C}(Y, \tau') \otimes \mathbf{C}(Z, \tau'')$  let*

$$\hat{F} : \mathbf{C}(X, \tau) \rightarrow \mathbf{C}(Y, \tau') \otimes \mathbf{C}(Z, \tau'')$$

*be defined by  $\hat{F}(a) = aF$ , for  $a \in \mathbf{C}(X, \tau)$ . Let*

$$(\mathbf{C}(Z, \tau''), \mathcal{C}, N)$$

*be the QSM space induced by QSM space  $(\mathbf{C}(X, \tau), \mathbf{Lip}(X, d_0), \|\cdot\|_{d_0})$  and quantum family of morphisms  $(\mathbf{C}(Z, \tau''), \hat{F})$ . Then the following are satisfied.*

- i)  $d_1(z, z') = \rho_N(\delta_z, \delta_{z'})$  for every  $z, z' \in Z$ .
- ii)  $\mathcal{C} \subset \mathbf{Lip}(Z, d_1)$ .
- iii)  $\|\cdot\|_{d_1} \leq N$ .

*Proof.* i) Let  $L = \|\cdot\|_{d_0}$ . Let us recall the definition of  $(\mathbf{C}(Z, \tau''), \mathcal{C}, N)$ . Let  $d$  be the semi metric on  $S(\mathbf{C}(Z, \tau''))$  defined by

$$d(\nu, \nu') = \sup\{\rho_L((\mu \otimes \nu)\hat{F}, (\mu \otimes \nu')\hat{F}) : \mu \in S(\mathbf{C}(Y, \tau'))\}.$$

Then  $N = L_d$  and  $\mathcal{C}$  is the space of all  $c \in \mathbf{C}(Z, \tau'')$  such that the map  $\nu \mapsto \nu(c)$  on  $S(\mathbf{C}(Z, \tau''))$  is continuous with  $d$  and  $N(c) < \infty$ . By Lemma 4.1, we have,

$$(5) \quad d(\delta_z, \delta_{z'}) = \rho_N(\delta_z, \delta_{z'}),$$

for every  $z, z' \in Z$ . Now, we explain the relation between  $d_1$  and  $d$ .

Let  $z, z' \in Z$  and  $y \in Y$ . Then

$$(\delta_y \otimes \delta_z)\hat{F} = \delta_{F(y, z)} \quad \text{and} \quad (\delta_y \otimes \delta_{z'})\hat{F} = \delta_{F(y, z')}.$$

On the other hand, by Proposition 2.2, for every  $x, x' \in X$ ,  $d_0(x, x') = \rho_L(\delta_x, \delta_{x'})$ . Thus

$$\rho_L((\delta_y \otimes \delta_z)\hat{F}, (\delta_y \otimes \delta_{z'})\hat{F}) = d_0(F(y, z), F(y, z')).$$

This formula together with the definitions of  $d$  and  $d_1$ , show that

$$(6) \quad d_1(z, z') \leq d(\delta_z, \delta_{z'}).$$

Let  $\mu \in S(\mathbf{C}(Y, \tau'))$  be arbitrary. We consider  $\mu$  as a probability Borel regular measure on  $(Y, \tau')$ . Then for every  $a \in \mathbf{Lip}(X, d_0)$  with  $\|a\|_{d_0} \leq 1$ , we have,

$$(7) \quad \begin{aligned} |(\mu \otimes \delta_z)\hat{F}(a) - (\mu \otimes \delta_{z'})\hat{F}(a)| &= \left| \int_Y (aF(y, z) - aF(y, z'))d_\mu(y) \right| \\ &\leq \int_Y |a(F(y, z)) - a(F(y, z'))|d_\mu(y). \end{aligned}$$

For every  $y \in Y$ , by Lemma 2.1,

$$|a(F(y, z)) - a(F(y, z'))| \leq d_0(F(y, z), F(y, z')).$$

Therefore, we have

$$(8) \quad |a(F(y, z)) - a(F(y, z'))| \leq d_1(z, z').$$

(8) and (7) implies that

$$|(\mu \otimes \delta_z)\hat{F}(a) - (\mu \otimes \delta_{z'})\hat{F}(a)| \leq d_1(z, z').$$

Therefore, by the definition of  $d$ ,

$$(9) \quad d(\delta_z, \delta_{z'}) \leq d_1(z, z').$$

Now, by (9) and (6),  $d(\delta_z, \delta_{z'}) = d_1(z, z')$ , and thus by (5),

$$d_1(\delta_z, \delta_{z'}) = \rho_N(\delta_z, \delta_{z'})$$

for every  $z, z' \in Z$ , and i) is satisfied. ii) and iii) are immediate consequence of i) and definitions of  $\mathcal{C}$ ,  $\|\cdot\|_{d_1}$  and  $N$ .  $\square$

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